

# Parafermions in the $\tau_2$ model II

Helen Au-Yang and Jacques H H Perk

Department of Physics, Oklahoma State University,  
145 Physical Sciences, Stillwater, OK 74078-3072, USA

E-mail: perk@okstate.edu, helenperk@yahoo.com

**Abstract.** Many years ago Baxter introduced an inhomogeneous two-dimensional classical spin model, now called the  $\tau_2(t)$  model with free boundary conditions, and he specialized the resulting quantum spin-chain Hamiltonian in a special limit to a simple clock Hamiltonian. Recently, Fendley showed that this clock Hamiltonian can be expressed in terms of free “parafermions.” Baxter followed this up by showing that this construction generalizes to the more general  $\tau_2(t)$  model, provided some conjectures hold. In this paper, we will compare the different notations and approaches enabling us to express the Hamiltonians in terms of projection operators as introduced by Fendley. By examining the properties of the raising operators, we are then able to prove the last unproven conjecture in Baxter’s paper left in our previous paper. Thus the eigenvectors can all be written in terms of these raising operators.

## 1. Introduction

In his study of parafermionic spin chains [1, 2], Fendley was led to consider the simple open spin-chain Hamiltonian introduced by Baxter [3, 4],

$$\mathcal{H} = - \sum_{j=1}^L \alpha_j \mathbf{X}_j - \sum_{j=1}^{L-1} \gamma_j \mathbf{Z}_j \mathbf{Z}_{j+1}^{-1}, \quad (1)$$

which is also equation (B1.5) in [5].<sup>‡</sup> Here  $\mathbf{X}_j$  and  $\mathbf{Z}_j$  on site  $j$  of the chain are copies of the  $N$ -by- $N$  matrices,

$$[\mathbf{X}]_{\sigma,\sigma'} = \delta(\sigma, \sigma' + 1), \quad [\mathbf{Z}]_{\sigma,\sigma'} = \omega^\sigma \delta(\sigma, \sigma'), \quad \mathbf{Z}\mathbf{X} = \omega\mathbf{X}\mathbf{Z}, \quad \mathbf{Z}^N = \mathbf{X}^N = \mathbf{1}, \quad (2)$$

generalizing the Pauli matrices  $\sigma^x$  and  $\sigma^z$  to  $N > 2$  and called  $\tau$  and  $\sigma$  in [1, 2]. Also, in (2) we have  $\omega = e^{2\pi i/N}$  and  $\sigma, \sigma' = 0, \dots, N-1$  in  $\mathbb{Z}_N$ .

In [2] Fendley succeeded in constructing operators generating the free parafermions associated with (1), upon which Baxter generalized [5] this construction to the full inhomogeneous  $\tau_2$  model with free boundary conditions from which Hamiltonian (1) was derived by him [3, 4, 7]. This generalization is of interest as the  $\tau_2$  model is an intermediate [8, 9] between the six-vertex model and the integrable chiral Potts model [10, 11, 12].

<sup>‡</sup> Equations in [5] are denoted here by prefacing B to the equation number; those from [2] by adding F in front of their number, and we preface AP to the equation numbers in [6].

Comparing (1) with (F34) in [2], we find not only an overall sign difference, but also the change of  $\mathbf{Z} \rightarrow \mathbf{Z}^{-1}$  corresponding to a left-right reflection of the spin chain. This is because we are following the conventions of Baxter in [5], just as we did in our previous paper [6], which the current paper follows up proving the final conjecture in [5] not resolved in [6].§

### 1.1. The $\tau_2$ model with open boundaries and corresponding Hamiltonian

The  $\tau_2(t)$  model with cyclic boundary conditions is defined by Baxter in (B2.6) [5] through the transfer matrix between two rows with spins  $\sigma_0, \dots, \sigma_L$  and  $\sigma'_0, \dots, \sigma'_L$ ,

$$\tau_2(t)_{\sigma, \sigma'} = \prod_{j=0}^L W_j(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j), \quad (\sigma_{L+1} \equiv \sigma_0, \quad \sigma'_{L+1} \equiv \sigma'_0), \quad (3)$$

where the nonzero Interaction-Round-a-Face (IRF) weights are given in (B2.3)|| as

$$\begin{aligned} W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j) &= b_{2j-1}b_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1} t c_{2j-1}c_{2j}, \\ W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j - 1) &= -\omega t d_{2j-1}b_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1} t a_{2j-1}c_{2j}, \\ W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j) &= b_{2j-1}d_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1} c_{2j-1}a_{2j}, \\ W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j - 1) &= -\omega t d_{2j-1}d_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1} a_{2j-1}a_{2j}. \end{aligned} \quad (4)$$

The transfer matrices  $\tau_2(t)$  form a commuting family parametrized by  $t$ , irrespective of the choice of the inhomogeneous constants  $a_j, b_j, c_j, d_j$ , which are periodic modulo  $2L + 2$  in  $j$ , but do not have to satisfy the chiral Potts curve relations (9) in [11].

In [7, eq. 73] and in (B3.1) Baxter chose  $a_{-1} = d_{-1} = 0$ . Then, as seen from (4),  $W_0(\sigma_0, \sigma_1, \sigma'_1, \sigma'_0) = 0$  if  $\sigma_0 \neq \sigma'_0$ . Therefore, because of periodic boundary conditions, we must have  $\sigma_0 \equiv \sigma_{L+1} \equiv \sigma'_0 \equiv \sigma'_{L+1}$ . Also, using the functional equations [7, eq. 47] or (B2.12), Baxter derived

$$\tau_2(t)\tau_2(\omega t) \cdots \tau_2(\omega^{N-1}t) = f(t^N)\mathbf{1}, \quad (5)$$

with  $f(x)$  some polynomial and  $\mathbf{1}$  the unit matrix of dimension  $N^{L+1}$ . This last statement follows as  $z(t)$  in (B2.13) now vanishes,  $\tau_2(t)$  is a polynomial in  $t$  with matrix coefficients and (5) is invariant under  $t \rightarrow \omega t$ .

Baxter specialized further to  $a_{-1} = d_{-1} = c_{-1} = c_{2L} = 0$ , and  $b_{-1} = b_{2L} = 1$ , see (B3.1) and (B3.4).¶ Then the relevant boundary weights become

$$\begin{aligned} W_0(\sigma_0, \sigma_1, \sigma_1, \sigma_0) &= b_0, & W_0(\sigma_0, \sigma_1, \sigma_1, \sigma_0 - 1) &= 0, \\ W_0(\sigma_0, \sigma_1, \sigma_1 - 1, \sigma_0) &= d_0, & W_0(\sigma_0, \sigma_1, \sigma_1 - 1, \sigma_0 - 1) &= 0, \\ W_L(\sigma_L, \sigma_0, \sigma_0, \sigma_L) &= b_{2L-1}, & W_L(\sigma_L, \sigma_0, \sigma_0, \sigma_L - 1) &= -\omega t d_{2L-1}. \end{aligned} \quad (6)$$

§ To facilitate comparisons with and between the cited papers, we shall outline the differences in notations and approaches, while also mentioning equivalences between equations. For more historical context and citations on parafermions we refer to the introduction of [6].

|| For the proofs in [6] we found it more convenient to use the equivalent vertex model formulation, see figure 5 in [7]. The IRF formulation used in (3) and (4) here corresponds to figure 4 in [7].

¶ Here we did not set  $b_j = 1$  for  $0 \leq j \leq 2L - 1$ . As the weights  $W_L$  with  $a_{2L}$  and  $d_{2L}$  now do not show up in  $\tau_2(t)$ , we can set  $a_{2L} = d_{2L} = 0$  also. The resulting  $\tau_2(t)$  is homogeneous in all its  $a, b, c, d$ .

The two weights  $W_L$  in (4) with  $\sigma_0 \neq \sigma'_0$  play no role as they are always paired with a vanishing  $W_0 \equiv W_{L+1}$  weight. Therefore, from (6) one concludes that  $\tau_2(t)$  does not depend on  $\sigma_0$  and  $\sigma'_0$ . Choosing  $\sigma_0 = \sigma'_0 = 0$  we reduce  $\tau_2(t)$  to become a  $N^L$ -by- $N^L$  matrix. This is how Baxter in [5] made it to be the transfer matrix of a model with free boundary spins at  $j = 1$  and  $j = L$ .<sup>+</sup> From (4) and (6) with  $W_0$  independent of  $t$ , it follows that  $\tau_2(t)$  is a polynomial of degree  $L$  in  $t$ ,

$$\tau_2(t) = \sum_{m=0}^L (\omega t)^m \tau_{2,m}, \quad \tau_{2,0} = \tau_2(0) = A_0 \mathbf{1}, \quad A_0 \equiv \prod_{\ell=0}^{2L-1} b_\ell. \quad (7)$$

Therefore, assuming all  $b_\ell \neq 0$ , the following expansion\* in powers of  $t$ ,

$$t \frac{d}{dt} \ln \tau_2(t) = \sum_{m=1}^{\infty} (\omega t)^m \mathcal{H}^{(m)}, \quad \tau_2(t) = A_0 \exp \left( \sum_{m=1}^{\infty} \frac{(\omega t)^m}{m} \mathcal{H}^{(m)} \right), \quad (8)$$

exists for the inhomogeneous  $\tau_2(t)$  model, with the leading term giving the Hamiltonian  $\mathcal{H} = \mathcal{H}^{(1)}$  and all  $\mathcal{H}^{(m)}$  constituting an infinite set of commuting Hamiltonians. Baxter gave the explicit form of  $\mathcal{H} = A_0^{-1} \tau_{2,1}$  in (B3.22) using the normalization  $b_j \equiv 1$ .<sup>‡</sup>

From (5) and (7), we conclude that

$$\tau_2(t) \tau_2(\omega t) \cdots \tau_2(\omega^{N-1} t) = A_0^N \mathbf{1} \prod_{j=1}^L (1 - r_j^N t^N), \quad (9)$$

where the  $L$  parameters  $r_j$  are the roots of a degree  $NL$  polynomial (B3.16),

$$s_0 r_j^{NL} + s_1 r_j^{N(L-1)} + s_2 r_j^{N(L-2)} + \cdots + s_L = 0, \quad \text{for } j = 1, \dots, L. \quad (10)$$

Thus Baxter obtained all the eigenvalues of the  $\tau_2(t)$  matrix, namely

$$\tau_2(t) = A_0 \prod_{j=1}^L (1 - r_j \omega^{1+p_j} t), \quad 0 \leq p_j \leq N-1, \quad 1 \leq j \leq L. \quad (11)$$

Consequently we have from (8) also all the  $NL$  eigenvalues of the higher Hamiltonians,

$$-\mathcal{H}^{(m)} |p_1, \dots, p_L\rangle = \sum_{j=1}^L (r_j \omega^{p_j})^m |p_1, \dots, p_L\rangle. \quad (12)$$

with  $|p_1, \dots, p_L\rangle$  denoting the corresponding eigenvector. In section 2 we will discuss what Fendley [2] did to express such matrices in terms of projection operators.

<sup>+</sup> In Appendix B.2 of [6], we showed that the object constructed by Fendley in (F50) is identical to this  $\tau_2$  in the clock model limit.

<sup>\*</sup> Here  $\mathcal{H}^{(j)} = -\mathbf{H}^{(j)}$  of (F48), not to be confused with what is defined in (F81) and (F82).

<sup>‡</sup> From (4) and (6) one sees that the  $\tau_{2,1}$  in (7) allows at most one single block of sites with  $\sigma'_\ell = \sigma_\ell - 1$ , as this block can only end with a weight  $W_k$  linear in  $t$ , forcing all other  $W_\ell$  to be constant. In analogy with [13, 14] each term in  $\tau_{2,1}$  has two factors, one with horizontal interaction proportional to some  $\mathbf{Z}_j \mathbf{Z}_k^{-1}$ , followed by the vertical spin flip  $\prod \mathbf{X}_\ell$  for the entire block. More explicitly, to get  $\tau_{2,1}$ , one replaces in  $\tau_{2,0}$  on the left either one  $b_{2j}$  by  $d_{2j} \mathbf{X}_{j+1}$  or one  $b_{2j-1}$  by  $-c_{2j-1} \mathbf{Z}_j$ , and on the right either one  $b_{2k-1}$  by  $-d_{2k-1}$  or one  $b_{2k}$  by  $c_{2k} \mathbf{Z}_{k+1}^{-1}$ . All intermediate  $b_{2\ell-1}$  are replaced by  $a_{2\ell-1} \mathbf{Z}_\ell$  and the  $b_{2\ell}$  by  $\omega a_{2\ell} \mathbf{Z}_{\ell+1}^{-1} \mathbf{X}_{\ell+1}$ . Collecting all  $\mathbf{X}_\ell$  in the second factor, this is how one can recover (B3.22).

### 1.2. Generalized Jordan–Wigner transform

Because of the difference of conventions between [2] and [5], we have to modify the basic parafermion operators  $\psi_\ell$  defined in (F37). First let us define, in addition to the  $\mathbf{X}_j$  and  $\mathbf{Z}_j$  defined through (2), the operators  $\mathbf{Y}_j$  as copies of  $\mathbf{Y}$  on sites  $j$ , where

$$\mathbf{Y} \equiv \omega^{(N-1)/2} \mathbf{X}^{-1} \mathbf{Z} = \omega^{(N+1)/2} \mathbf{Z} \mathbf{X}^{-1}, \quad \mathbf{Y}^{-1} = \omega^{(1-N)/2} \mathbf{Z}^{-1} \mathbf{X}, \quad \mathbf{Y}^N = \mathbf{1}. \quad (13)$$

The scalar factors arise, as in the evaluation of  $\mathbf{Y}^N$  we have to commute  $\mathbf{X}^{-1}$  and  $\mathbf{Z}$  exactly  $\frac{1}{2}N(N-1)$  times. We define the basic parafermions as

$$\psi_{2j-2} = \left( \prod_{\ell=1}^{j-1} \mathbf{X}_\ell \right) \mathbf{Z}_j^{-1}, \quad \psi_{2j-1} = \left( \prod_{\ell=1}^{j-1} \mathbf{X}_\ell \right) \mathbf{Y}_j^{-1}, \quad \psi_0 = \Gamma_0 = \mathbf{Z}_1^{-1}, \quad (14)$$

for  $1 \leq j \leq L$ . From the commutation relations of  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{Z}$ , it follows that

$$\psi_j \psi_k = \omega^{-1} \psi_k \psi_j \quad \text{for } j < k, \quad \psi_j^N = \mathbf{1}. \quad (15)$$

If  $N = 2$ ,  $\omega = -1$ ,  $\omega^{(N-1)/2} = i$ , and  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{Z}$  become the Pauli matrices  $\sigma^x$ ,  $\sigma^y$  and  $\sigma^z$ , which are equal to their own inverses. Then the  $\psi_\ell$  become the  $\Gamma_\ell$  of Kaufman [14].

Hamiltonian (1) is a special case of (B3.23) in [5], which we rewrite using (13) as

$$\begin{aligned} \mathcal{H} = & - \sum_{j=1}^L \sum_{k=j}^L \omega^{k-j+(N-1)/2} \frac{d_{2j-2}}{b_{2j-2}} \left( \prod_{\ell=2j-1}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} \mathbf{Z}_j \left( \prod_{\ell=j}^{k-1} \mathbf{X}_\ell \right) \mathbf{Y}_k^{-1} \\ & + \sum_{j=1}^{L-1} \sum_{k=j+1}^L \omega^{k-j-1} \frac{c_{2j-1}}{b_{2j-1}} \left( \prod_{\ell=2j}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} \mathbf{Y}_j \left( \prod_{\ell=j}^{k-1} \mathbf{X}_\ell \right) \mathbf{Y}_k^{-1} \\ & - \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j-(N+1)/2} \frac{c_{2j-1}}{b_{2j-1}} \left( \prod_{\ell=2j}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} \mathbf{Y}_j \left( \prod_{\ell=j}^k \mathbf{X}_\ell \right) \mathbf{Z}_{k+1}^{-1} \\ & + \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j} \frac{d_{2j-2}}{b_{2j-2}} \left( \prod_{\ell=2j-1}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} \mathbf{Z}_j \left( \prod_{\ell=j}^k \mathbf{X}_\ell \right) \mathbf{Z}_{k+1}^{-1}. \end{aligned} \quad (16)$$

For the special case  $N = 2$ , after rotating  $\mathbf{Z}_\ell \rightarrow \sigma_\ell^x$ ,  $\mathbf{X}_\ell \rightarrow -\sigma_\ell^z$  and  $\mathbf{Y}_\ell \rightarrow \sigma_\ell^y$ , we recognize a generalized XY-model, like the spin-chain Hamiltonian that Suzuki introduced [16, 17] to commute with the transfer matrix of the dimer model.

Hamiltonian (16) may be expressed in terms of the parafermions (14) as

$$\begin{aligned} \mathcal{H} = & - \sum_{j=1}^L \sum_{m=j}^L \omega^{m-j+(N-1)/2} \left( \prod_{\ell=2j-1}^{2m-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2j-2} d_{2m-1}}{b_{2j-2} b_{2m-1}} \psi_{2j-2}^{-1} \psi_{2m-1} \\ & - \sum_{j=1}^{L-1} \sum_{m=j}^{L-1} \omega^{m-j} \left[ \omega^{-(N+1)/2} \left( \prod_{\ell=2j}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2j-1} c_{2m}}{b_{2j-1} b_{2m}} \psi_{2j-1}^{-1} \psi_{2m} \right. \\ & \left. - \left( \prod_{\ell=2j-1}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{d_{2j-2} c_{2m}}{b_{2j-2} b_{2m}} \psi_{2j-2}^{-1} \psi_{2m} - \left( \prod_{\ell=2j}^{2m} \frac{a_\ell}{b_\ell} \right) \frac{c_{2j-1} d_{2m+1}}{b_{2j-1} b_{2m+1}} \psi_{2j-1}^{-1} \psi_{2m+1} \right], \end{aligned} \quad (17)$$

setting  $k = m + 1$  in the second term of (16) and  $k = m$  in the other three. Thus we find that  $\mathcal{H}$  is quadratic in the parafermions, just like operators in the Onsager algebra

[13] and the generalized XY-model are quadratic in fermions [15]. If, as Baxter did in (B3.25), we set  $a_j = 0$  for  $j = 1, \dots, 2L - 2$  in (17), only the terms with  $m = j$  in the first two lines of (17) survive, the empty products over  $\ell$  being equal to 1. Then (17) reduces to the Hamiltonian below (F37) in [2] corresponding to (F34), which is also Baxter's special clock Hamiltonian (1).

### 1.3. Raising Operators

Inspired by Fendley's paper [2], Baxter defined in (B4.2)

$$\mathbf{\Gamma}_0 = \mathbf{Z}_1^{-1}, \quad \mathbf{\Gamma}_{j+1} = (\omega^{-1} - 1)^{-1}(\mathcal{H}\mathbf{\Gamma}_j - \mathbf{\Gamma}_j\mathcal{H}), \quad (j \geq 0), \quad (18)$$

which is almost the same as (F80). Using  $\mathbf{\Gamma}_0 = \boldsymbol{\psi}_0$ , (15) and (17), it is straightforward to show that

$$\mathbf{\Gamma}_1 = \frac{d_0}{b_0} \left[ \sum_{m=1}^L \omega^{m+(N-1)/2} \left( \prod_{\ell=1}^{2m-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2m-1}}{b_{2m-1}} \boldsymbol{\psi}_{2m-1} - \sum_{m=1}^{L-1} \omega^m \left( \prod_{\ell=1}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2m}}{b_{2m}} \boldsymbol{\psi}_{2m} \right], \quad (19)$$

which is rather complicated. Nevertheless, using (15) again, we can easily show

$$\mathbf{\Gamma}_0 \mathbf{\Gamma}_1 = \omega^{-1} \mathbf{\Gamma}_1 \mathbf{\Gamma}_0. \quad (20)$$

Based on numerical evidence, Baxter found that the infinite sequence of the  $\mathbf{\Gamma}_j$  truncates, as he conjectured that the  $\mathbf{\Gamma}$  matrices satisfy the equation

$$s_0 \mathbf{\Gamma}_{NL+j} + s_1 \mathbf{\Gamma}_{N(L-1)+j} + \dots + s_L \mathbf{\Gamma}_j = 0, \quad \text{for } j = 0, \quad (21)$$

with the same  $s_\ell$  as in (10), which is also (B3.16). In [6], we have proven that this equation holds for any nonnegative  $j$ . Fendley on the other hand introduced a different basis in (F84) by making certain subtractions so that the iteration terminates as shown in (F88). Since the Hamiltonian (17) is more complicated than (1), his method may not work in the general case.

We can use (18) to express the commutators  $[\mathcal{H}, \mathbf{\Gamma}_j]$  in terms of  $\mathbf{\Gamma}_{j+1}$  for  $0 \leq j < NL$  and use (21) to eliminate  $\mathbf{\Gamma}_{NL}$ . Thus we recover (B4.11),

$$\mathcal{H}\mathbf{\Gamma}_j - \mathbf{\Gamma}_j\mathcal{H} = (\omega^{-1} - 1) \sum_{k=0}^{NL-1} h_{jk} \mathbf{\Gamma}_k, \quad (22)$$

where

$$\begin{aligned} h_{ij} &= \delta_{i+1,j}, \quad (0 \leq i \leq NL-2, 0 \leq j \leq NL-1); \\ h_{NL-1,mN} &= -s_{L-m}/s_0, \quad (0 \leq m \leq L-1), \\ h_{NL-1,j} &= 0, \quad (j \not\equiv 0 \pmod{N}). \end{aligned} \quad (23)$$

In (B4.10), Baxter denoted this  $NL \times NL$  matrix with elements  $h_{jk}$  by  $\mathbf{H}$ , (not to be confused with the Hamiltonian  $\mathcal{H}$ ), and he showed that its characteristic polynomial is

$$|\mathbf{H} - \lambda \mathbf{I}| = s_0 \lambda^{NL} + s_1 \lambda^{N(L-1)} + \dots + s_{L-1} \lambda^N + s_L. \quad (24)$$

Comparing with (10), we find the  $NL$  roots to be  $\lambda_i = r_k \omega^p$ , with  $1 \leq k \leq L$  and  $0 \leq p \leq N-1$ .

In section 5 of [6], we calculated the eigenvectors of  $\mathbf{H}$ , which form the columns of the matrix  $\mathbf{P}$  diagonalizing  $\mathbf{H}$ ,

$$\mathbf{P}^{-1}\mathbf{H}\mathbf{P} = \mathbf{H}_d, \quad (25)$$

and we found that  $\mathbf{P}$  is the Vandermonde matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{NL} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{NL}^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \cdots & \lambda_{NL}^3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{NL-1} & \lambda_2^{NL-1} & \lambda_3^{NL-1} & \cdots & \lambda_{NL}^{NL-1} \end{bmatrix}. \quad (26)$$

In order to be consistent with notations in [5] we choose its matrix indices as follows,

$$P_{ij} = \lambda_j^i, \quad \text{with } 0 < i < NL - 1, 1 < j < NL. \quad (27)$$

According to Prony's 1795 result [18, 19, 20], the elements  $(\mathbf{P}^{-1})_{jk}$  of its inverse are the coefficients of the polynomials  $f_j(z)$  given by

$$f_j(z) = \prod_{i=1, i \neq j}^{NL} \frac{z - \lambda_i}{\lambda_j - \lambda_i} = \sum_{k=0}^{NL-1} (P^{-1})_{jk} z^k, \quad \text{satisfying } f_j(\lambda_i) = \delta_{ji}. \quad (28)$$

In (B4.17) Baxter then defined new raising operators,

$$\hat{\Gamma}_i = \sum_{j=0}^{NL-1} P_{ij}^{-1} \Gamma_j \quad (29)$$

so that (22) becomes (B4.18),

$$\mathcal{H} \hat{\Gamma}_j - \hat{\Gamma}_j \mathcal{H} = (\omega^{-1} - 1) \lambda_j \hat{\Gamma}_j. \quad (30)$$

## 2. Projection operators

We have shown in appendix B of [6] that the inverse Vandermonde in (28) is related to the inverse of Fendley's Vandermonde matrix  $\mathcal{X}$  on page 28 of [2], namely

$$P_{i, \ell N + q}^{-1} = P_{(p, k), \ell N + q}^{-1} = \frac{1}{N} (\mathcal{X}^{-1})_{k, \ell} (r_k \omega^p)^{-q}, \quad i = kN + p, \quad (31)$$

which is (AP B.11) in [6]. Consequently, combining (F100) and (F103) and generalizing the result to our  $\tau_2$  case, we find that the projection operator is

$$\mathcal{P}_{\omega^p, k} = - \sum_{\ell=0}^{L-1} \sum_{q=0}^{N-1} P_{p, k; \ell N + q}^{-1} \mathcal{H}^{(\ell N + q)}, \quad (32)$$

where we use  $\mathcal{P}$  for the projection operator in order to distinguish it from the Vandermonde matrix  $\mathbf{P}$ . Using

$$\mathbf{P} \cdot \mathbf{P}^{-1} = 1, \quad (33)$$

where  $\mathbf{P}$  is given in (26), and letting  $\lambda_i = r_k \omega^p$  and  $i = kN + p$ , we find

$$\sum_{i=1}^{NL} \lambda_i^m P_{i, \ell N + q}^{-1} = \sum_{k=1}^L \sum_{p=0}^{N-1} (r_k \omega^p)^m P_{(p,k), \ell N + q}^{-1} = \delta_{m, \ell N + q}. \quad (34)$$

Multiplying (32) by  $\lambda_i^m$  and summing over all  $i$ , we find

$$\mathcal{H}^{(m)} = - \sum_{k=1}^L \sum_{p=0}^{N-1} (r_k \omega^p)^m \mathcal{P}_{\omega^p, k}, \quad (35)$$

which is the same as (F105), but now generalized to the full  $\tau_2(t)$  model with free boundaries. Since the  $\mathcal{H}^{(m)}$  commute with one another, the projection operators in (32) must also. Thus

$$[\mathcal{P}_{\omega^p, k}, \mathcal{P}_{\omega^q, \ell}] = 0. \quad (36)$$

In (12) we introduced the basis of  $N^L$  eigenstates  $\{|n_1, n_2, \dots, n_L\rangle, (n_j \in \mathbb{Z}_N)\}$  on which the  $\mathcal{H}^{(m)}$  act as

$$\mathcal{H}^{(m)} |n_1, n_2, \dots, n_L\rangle = - \sum_{k=1}^L (r_k \omega^{n_k})^m |n_1, n_2, \dots, n_L\rangle. \quad (37)$$

Substituting (35) into (37), we find that

$$\mathcal{P}_{\omega^p, k} |n_1, n_2, \dots, n_L\rangle = \delta_{p, n_k} |n_1, n_2, \dots, n_L\rangle, \quad (38)$$

which shows that

$$\mathcal{P}_{\omega^p, k}^2 = \mathcal{P}_{\omega^p, k}, \quad \mathcal{P}_{\omega^p, k} \mathcal{P}_{\omega^q, k} = \delta_{p, q} \mathcal{P}_{\omega^p, k}, \quad \sum_{p=0}^{N-1} \mathcal{P}_{\omega^p, k} = \mathbf{1}. \quad (39)$$

These properties show that the  $\mathcal{P}_{\omega^p, k}$  are indeed projection operators, agreeing with what Fendley found in [2] for the special case (1). Next we set  $\lambda_j = r_\ell \omega^q$  and  $j = \ell N + q$  in (30), so that  $\hat{\Gamma}_j \equiv \hat{\Gamma}_{q, \ell}$ . Then using (35) with  $m = 1$ , we have

$$\sum_{k=1}^L \sum_{p=0}^{N-1} (r_k \omega^p) [\mathcal{P}_{\omega^p, k} \hat{\Gamma}_{q, \ell} - \hat{\Gamma}_{q, \ell} \mathcal{P}_{\omega^p, k}] = r_\ell (\omega^{q-1} - \omega^q) \hat{\Gamma}_{q, \ell}. \quad (40)$$

This implies the relation,

$$[\mathcal{P}_{\omega^p, k} \hat{\Gamma}_{q, \ell} - \hat{\Gamma}_{q, \ell} \mathcal{P}_{\omega^p, k}] = \delta_{k, \ell} (\delta_{p, q-1} - \delta_{p, q}) \hat{\Gamma}_{q, \ell}, \quad (41)$$

in agreement with the equation above (F106) in [2]. In the derivation of (41) we used that  $\hat{\Gamma}_{q, \ell}$  only acts on the  $n_\ell$  in the eigenstate  $|n_1, \dots, n_L\rangle$ , as was shown by Baxter [5] using (B4.21) following from (B4.19),<sup>††</sup> which we can rewrite as

$$(1 - r_\ell \omega^q t) \tau_2(t) \hat{\Gamma}_{q, \ell} = (1 - r_\ell \omega^{q+1} t) \hat{\Gamma}_{q, \ell} \tau_2(t) \quad (42)$$

and from which (30) also follows as the first nontrivial term in the expansion in powers of  $\omega t$ . The extra  $t$ -dependence means that we can forget about complications due to accidental degeneracies. The ratio of the coefficients in (42) is the ratio of two unique

<sup>††</sup>Some steps in the derivation in [5] were found from numerical work and proved in [6].

eigenvalues of  $\tau_2(t)$ , so that  $\widehat{\Gamma}_{q,\ell}$  has its only one nonzero matrix element between the two corresponding eigenvectors [5, section 4.3], raising  $n_\ell = q - 1$  to  $q$ .

In terms of the above projection operators, and using their properties (38) and (39), we find from (11)

$$\tau_2(t) = A_0 \prod_{k=1}^L \prod_{p=0}^{N-1} (1 - r_k \omega^{1+p} t \mathcal{P}_{\omega^p, k}) = A_0 \prod_{k=1}^L \left( 1 - \omega t \sum_{p=0}^{N-1} r_k \omega^p \mathcal{P}_{\omega^p, k} \right). \quad (43)$$

This agrees with (AP75) in [6] only if one identifies

$$\mathbf{u}_k = \sum_{p=0}^{N-1} r_k \omega^p \mathcal{P}_{\omega^p, k}. \quad (44)$$

Finally, as shown in [5, 6], the only non-vanishing elements of  $\widehat{\Gamma}_{p,k}$  are

$$\begin{aligned} \langle n_1, \dots, \overset{k}{p}, \dots, n_L | \widehat{\Gamma}_{p,k} | n_1, \dots, \overset{k}{p-1}, \dots, n_L \rangle \\ = \langle n_1, \dots, \overset{k}{p}, \dots, n_L | \Gamma_0 | n_1, \dots, \overset{k}{p-1}, \dots, n_L \rangle, \end{aligned} \quad (45)$$

see (AP86) and (AP95) for example. Each eigenstate is represented by  $L$  integers  $n_1, \dots, n_k, \dots, n_L$  with  $n_k = 0, \dots, N-1$ . The raising operator  $\widehat{\Gamma}_{p,k}$  raises the value  $n_k$  by one (mod  $N$ ) if  $n_k = p-1$  leaving the other  $L-1$  integers unchanged; if  $n_k \neq p-1$ , it kills the eigenstate.

### 3. Proof of (B5.4) or (AP96)

However, there is a much easier way to prove (45) and to generalize it using the Vandermonde matrix (26). Let  $\{n_\ell\}_k$  be the set  $\{n'_\ell\}$  with  $n'_\ell = n_\ell$  for  $\ell \neq k$  and  $n'_k = n_k - 1 \pmod{N}$ . Then from Baxter's argument [5]—see text around (42)—we know that  $\langle \{n_\ell\} | \widehat{\Gamma}_i | \{n'_\ell\} \rangle = 0$ , if  $\{n'_\ell\} \neq \{n_\ell\}_i$ . Applying the Vandermonde matrix  $\mathbf{P}$ , we also find  $\langle \{n_\ell\} | \Gamma_j | \{n'_\ell\} \rangle = 0$  for all  $j$ , if  $\{n'_\ell\} \neq \{n_\ell\}_k$  for all  $k$ . More precisely,

$$\langle \{n_\ell\} | \Gamma_j | \{n_\ell\}_k \rangle = (\lambda_{kN+\ell})^j \langle \{n_\ell\} | \widehat{\Gamma}_{kN+\ell} | \{n_\ell\}_k \rangle, \quad \lambda_{kN+\ell} = r_k \omega^{n_\ell}. \quad (46)$$

Thus the  $\Gamma_j$  can only have elements corresponding to raising one  $n_\ell$  by 1, so that the only nonzero elements are

$$\langle n_1, \dots, n_\ell, \dots, n_L | \Gamma_j | n_1, \dots, n_\ell - 1, \dots, n_L \rangle \neq 0, \quad \ell = 1, \dots, L. \quad (47)$$

Unlike the raising operator  $\widehat{\Gamma}_{k,p}$  which only can raise  $n_k = p-1$  by one, we find  $\Gamma_j$  can raise any  $n_\ell$  by one for any  $\ell$ .

Now we use (20) to prove (AP96). We write

$$\begin{aligned} 0 &= \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q}, \dots, n_L | (\Gamma_0 \Gamma_1 - \omega^{-1} \Gamma_1 \Gamma_0) | n_1, \dots, \overset{k}{p-1}, \dots, \overset{\ell}{q-1}, \dots, n_L \rangle \\ &= \sum_{\{n'_i\}} \left[ \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q}, \dots, n_L | \Gamma_0 | \{n'_i\} \rangle \langle \{n'_i\} | \Gamma_1 | n_1, \dots, \overset{k}{p-1}, \dots, \overset{\ell}{q-1}, \dots, n_L \rangle \right. \\ &\quad \left. - \omega^{-1} \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q}, \dots, n_L | \Gamma_1 | \{n'_i\} \rangle \langle \{n'_i\} | \Gamma_0 | n_1, \dots, \overset{k}{p-1}, \dots, \overset{\ell}{q-1}, \dots, n_L \rangle \right]. \end{aligned} \quad (48)$$



From (47), we find only two possibilities for the summand to be nonvanishing: either  $n'_k = p - 1$  and  $n'_i = n_i$  for  $i \neq k$ , or  $n'_\ell = q - 1$  and  $n'_i = n_i$  for  $i \neq \ell$ . Furthermore, we find from (46) that,

$$\begin{aligned} & \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q} - \epsilon, \dots, n_L | \mathbf{\Gamma}_1 | n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q} - \epsilon, \dots, n_L \rangle \\ &= r_k \omega^p \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q} - \epsilon, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q} - \epsilon, \dots, n_L \rangle, \\ & \langle n_1, \dots, \overset{k}{p} - \epsilon, \dots, \overset{\ell}{q}, \dots, n_L | \mathbf{\Gamma}_1 | n_1, \dots, \overset{k}{p} - \epsilon, \dots, \overset{\ell}{q} - 1, \dots, n_L \rangle \\ &= r_\ell \omega^q \langle n_1, \dots, \overset{k}{p} - \epsilon, \dots, \overset{\ell}{q}, \dots, n_L | \mathbf{\Gamma}_1 | n_1, \dots, \overset{k}{p} - \epsilon, \dots, \overset{\ell}{q} - 1, \dots, n_L \rangle, \end{aligned} \quad (49)$$

for  $\epsilon = 0, 1$ . Substituting (49) into (48), we prove the identity in (AP96) and therefore also (B5.4), which generalizes (F111).

Finally, we can construct all the eigenvectors by applying the raising operators on the ‘ground state’ with  $n_1 = n_2 = \dots = n_L = 0$  and denoted by

$$|\{0\}\rangle = |0, 0, \dots, 0\rangle. \quad (50)$$

Let the ordered product of  $p$  raising operators  $\widehat{\mathbf{\Gamma}}_{q,k}$  in descending order of  $q$  be

$$\mathbf{\Theta}_{p,k} = \widehat{\mathbf{\Gamma}}_{p,k} \widehat{\mathbf{\Gamma}}_{p-1,k} \cdots \widehat{\mathbf{\Gamma}}_{2,k} \widehat{\mathbf{\Gamma}}_{1,k}, \quad \mathbf{\Theta}_{0,k} = \mathbf{1}. \quad (51)$$

Any eigenvectors can be obtained as

$$|\{n_i\}\rangle = |n_1, n_2, \dots, n_L\rangle = C(\{n_i\}) \mathbf{\Theta}_{n_1,1} \mathbf{\Theta}_{n_2,2} \cdots \mathbf{\Theta}_{n_L,L} |\{0\}\rangle. \quad (52)$$

Alternatively, we can use

$$\widehat{\mathbf{\Theta}}_{p,k} \equiv \left( \sum_{q=0}^{N-1} \widehat{\mathbf{\Gamma}}_{q,k} \right)^p, \quad \widehat{\mathbf{\Theta}}_{N,k} = \sum_{q=0}^{N-1} \widehat{\mathbf{\Gamma}}_{N+q,k} \widehat{\mathbf{\Gamma}}_{N-1+q,k} \cdots \widehat{\mathbf{\Gamma}}_{2+q,k} \widehat{\mathbf{\Gamma}}_{1+q,k}, \quad (53)$$

identifying  $\widehat{\mathbf{\Gamma}}_{N+q,k} \equiv \widehat{\mathbf{\Gamma}}_{q,k}$ . If  $\widehat{\mathbf{\Theta}}_{N,k} \propto \mathbf{1}$ , as in proposal (F108), we can call the  $\widehat{\mathbf{\Theta}}_{1,k}$  cyclic raising (or shift) operators. One may consult section 6.3 of [2] for further discussion related to the special Hamiltonian (1).

#### 4. Summary

In this paper we presented some new results for the inhomogeneous  $\tau_2$  model with open boundary conditions and its associated Hamiltonians. In section 1 we have given an introduction including several formulae from papers of Baxter [5], Fendley [2] and ourselves [6] that are needed to make the present paper somewhat self-contained. We added new details and discussions and we discussed the differences in notations and symbols between the papers stemming in part from differences in conventions between [2] and [5]. We reviewed the eigenvalue spectrum and quantum numbers, and also the two sets of operators  $\mathbf{\Gamma}_j$  and  $\widehat{\mathbf{\Gamma}}_j$ .

In (32) of section 2 we introduced the complete set of projection operators  $\mathcal{P}_{\omega^p,k}$  defined in terms of the higher Hamiltonians  $\mathcal{H}^{(m)}$ , in full analogy with (F100) and (F103) for the special clock model; only we used  $\mathcal{H}^{(m)}$  instead of the  $-\mathbf{H}^{(m)}$  of Fendley. As

a consequence, in (35) and (43), the Hamiltonians  $\mathcal{H}^{(m)}$  and the  $\tau_2(t)$  matrix are all expressed in terms of the projection operators.

We then showed in section 3, applying the Vandermonde matrix, that the elements of the operators  $\langle \{n'_i\} | \Gamma_j | \{n_i\} \rangle$  can be non-zero if and only if any one of  $L$  integers, say  $n_k$ , increase by one,  $n'_k = n_k + 1$ . Finally, we proved conjecture (B5.4) or equivalently (AP96), generalizing (F111) and giving us the commutation relation for  $\hat{\Gamma}_{p,k}$  and  $\hat{\Gamma}_{q,\ell}$  with  $k \neq \ell$ .

## References

- [1] Fendley P 2012 Parafermionic edge zero modes in  $\mathbb{Z}_n$ -invariant spin chains *J. Stat. Mech.* 2012 P11020 (25 pp) (arXiv:1209.0472)
- [2] Fendley P 2014 Free parafermions *J. Phys. A: Math. Theor.* **47** 075001 (42pp) (arXiv:1310.6049)
- [3] Baxter R J 1989 A simple solvable  $Z_N$  Hamiltonian *Phys. Lett. A* **140** 155–7
- [4] Baxter R J 1989 Superintegrable chiral Potts model: Thermodynamic properties, an “inverse” model, and a simple associated Hamiltonian *J. Stat. Phys.* **57** 1–39
- [5] Baxter R J 2014 The  $\tau_2$  model and parafermions *J. Phys. A: Math. Theor.* **47** 315001 (12pp) (arXiv:1310.7074)
- [6] Au-Yang H and Perk J H H 2014 Parafermions in the  $\tau_2$  model *J. Phys. A: Math. Theor.* **47** 315002 (19pp) (arXiv:1402.0061)
- [7] Baxter R J 2004 Transfer matrix functional relations for the generalized  $\tau_2(t_q)$  model *J. Stat. Phys.* **117** 1–25 (arXiv:cond-mat/0409493)
- [8] Bazhanov V V and Stroganov Yu G 1990 Chiral Potts model as a descendant of the six-vertex model *J. Stat. Phys.* **59** 799–817
- [9] Baxter R J, Bazhanov V V and Perk J H H 1990 Functional relations for transfer matrices of the chiral Potts model *Int. J. Mod. Phys. B* **4** 803–70
- [10] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M-L 1987 Commuting transfer matrices in the chiral Potts models: Solutions of the star-triangle equations with genus  $> 1$  *Phys. Lett. A* **123** 219–23
- [11] Baxter R J, Perk J H H and Au-Yang H 1988 New solutions of the star-triangle relations for the chiral Potts model *Phys. Lett. A* **128** 138–42
- [12] Perk J H H 2016 The early history of the integrable chiral Potts model and the odd-even problem *J. Phys. A: Math. Theor.* **49** 153001 (20pp) (arXiv:1511.08526)
- [13] Onsager L 1944 Crystal statistics. I. A two-dimensional model with an order-disorder transition *Phys. Rev.* **65** 117–49
- [14] Kaufman B 1949 Crystal statistics. II. Partition function evaluated by spinor analysis *Phys. Rev.* **76** 1232–43
- [15] Jha D K and Valatin J G 1973 XY model and algebraic methods *J. Phys. A: Math., Nucl. Gen.* **6** 1679–92
- [16] Suzuki M 1971 The dimer problem and the generalized XY model *Phys. Lett. A* **34** 338–9
- [17] Suzuki M 1971 Relationship among exactly soluble models of critical phenomena. I—2D Ising model, dimer problem and the generalized XY-model *Progr. Theor. Phys.* **46** 1337–57
- [18] Prony (G C F M R de) 1795 Considérations sur les principes de la méthode inverse des différences *J. de l'Éc. Polyt.* **1** (3) 209–73, see pp 264–5  
(Online at <http://gallica.bnf.fr/ark:/12148/bpt6k4336621/f23.image>)
- [19] Muir T 1906 *The Theory of Determinants in the Historical Order of Development* vol I (McMillan: London) pp 306–8
- [20] Au-Yang H and Perk J H H 2009 Eigenvectors in the superintegrable model II: ground-state sector *J. Phys. A: Math. Theor.* **42** 375208 (16pp) (arXiv:0803.3029)